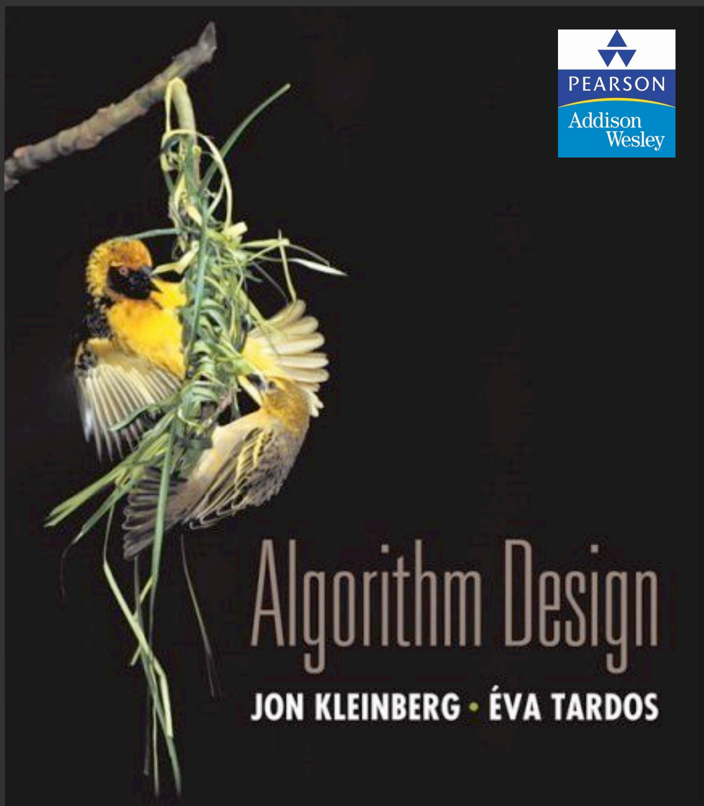
5. DIVIDE AND CONQUER I

- *mergesort*
- *counting inversions*
- *closest pair of points*



Last updated on Oct 2, 2013 9:51 AM

Divide-and-conquer paradigm

Divide-and-conquer.

- Divide up problem into several subproblems.
- Solve each subproblem recursively.
- Combine solutions to subproblems into overall solution.

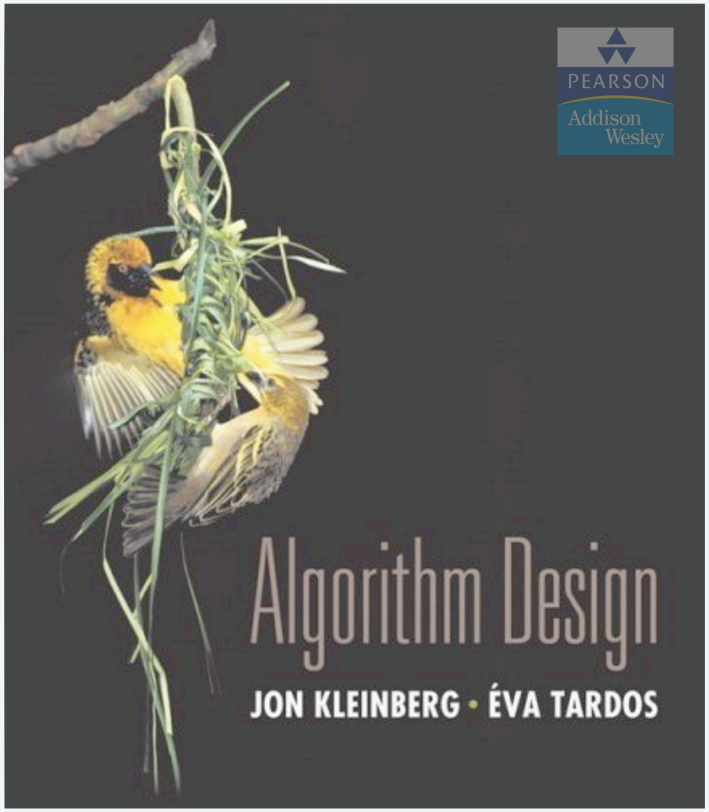
Most common usage.

- Divide problem of size n into two subproblems of size $n/2$ in linear time.
- Solve two subproblems recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: $\Theta(n^2)$.
- Divide-and-conquer: $\Theta(n \log n)$.

attributed to Julius Caesar



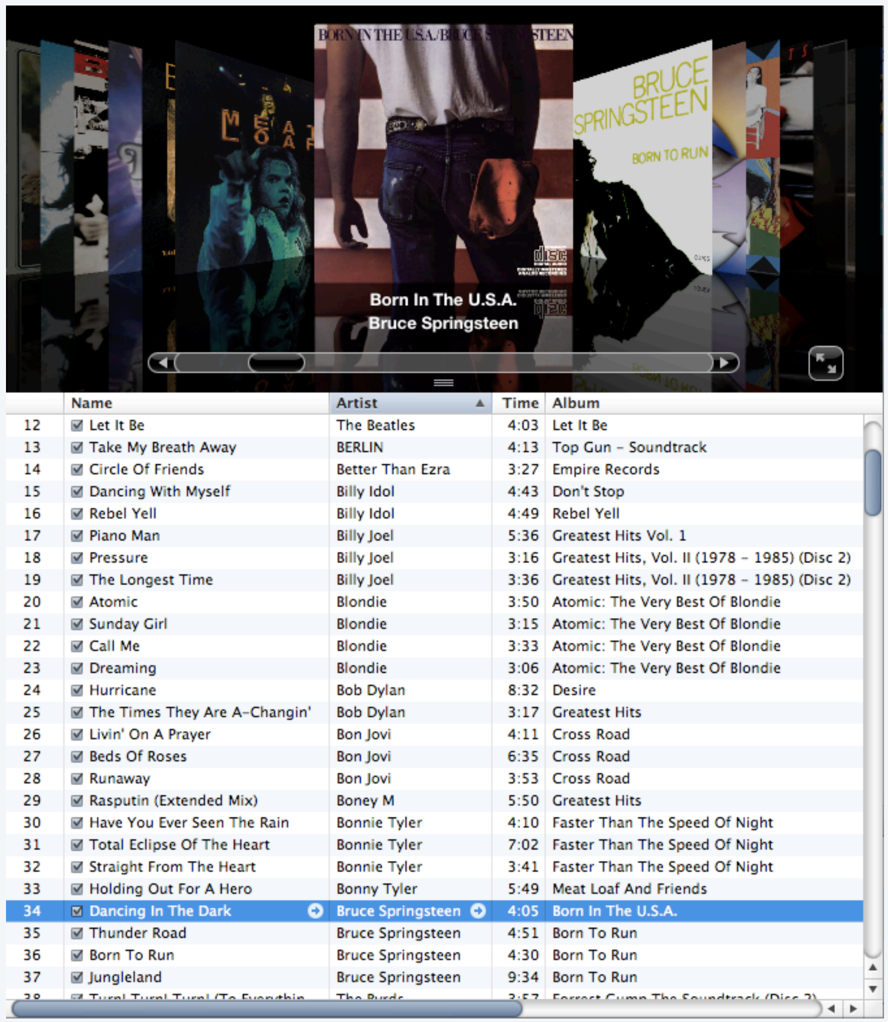
SECTION 5.1

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*

Sorting problem

Problem. Given a list of n elements from a totally-ordered universe, rearrange them in ascending order.



Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.

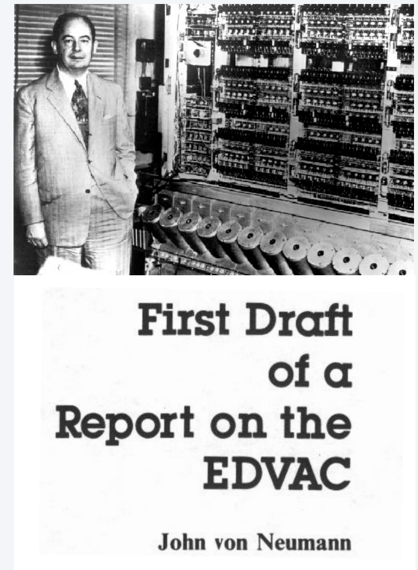
- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Minimum spanning trees (Kruskal's algorithm).
- Scheduling to minimize maximum lateness or average completion time.
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.



input

A	L	G	O	R	I	T	H	M	S

sort left half

A	G	L	O	R	I	T	H	M	S

sort right half

A	G	L	O	R	H	I	M	S	T

merge results

A	G	H	I	L	M	O	R	S	T

Merging

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).

A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of size $\leq n$.

Note. $T(n)$ is monotone nondecreasing.

Mergesort recurrence.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence.
Initially we assume n is a power of 2 and replace $\leq$ with $=$.

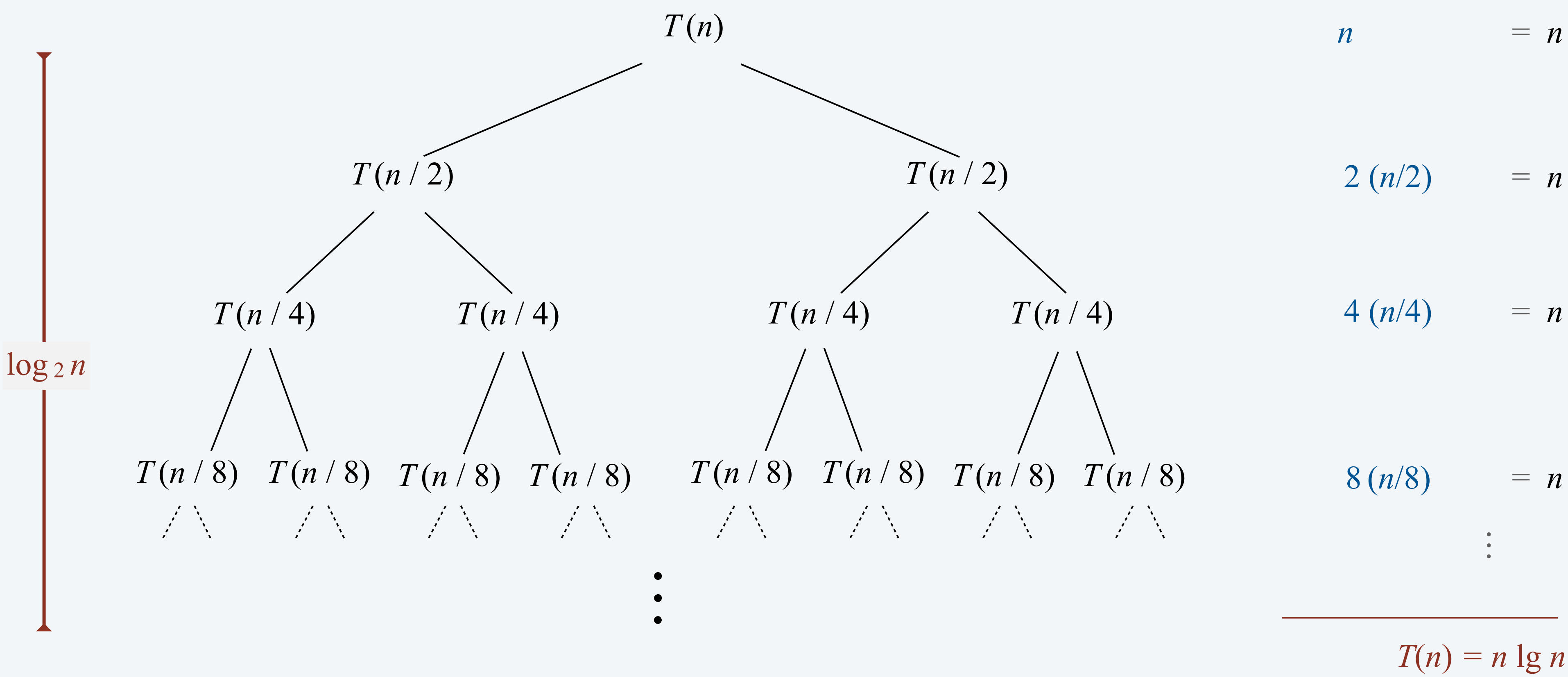
Divide-and-conquer recurrence: proof by recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 1.



Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

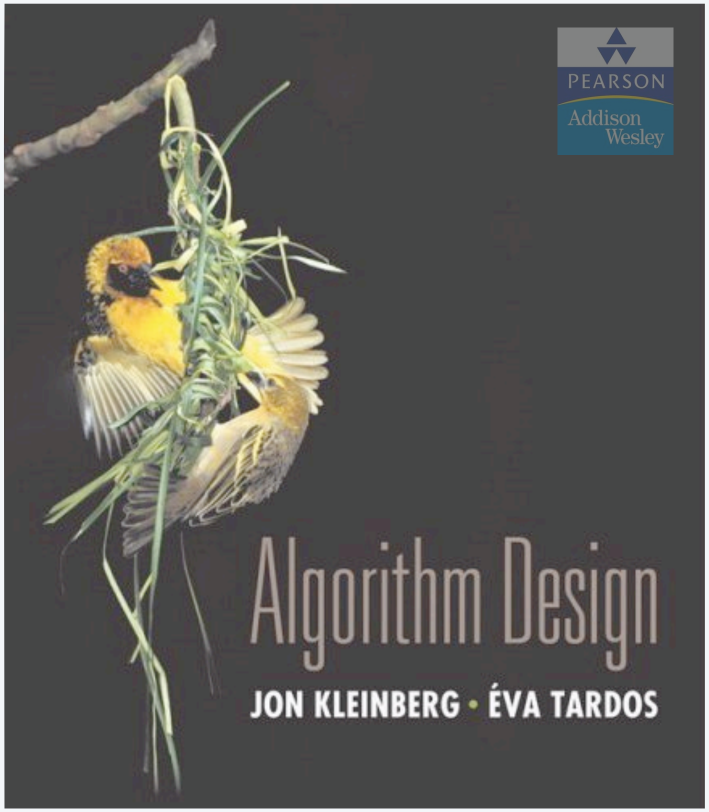
assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad \blacksquare \end{aligned}$$



SECTION 5.3

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*

Counting inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: $1, 2, \ldots, n$.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 inversions: 3–2, 4–2

Brute force: check all $\Theta(n^2)$ pairs.

Counting inversions: applications

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's tau distance).

Rank Aggregation Methods for the Web

Cynthia Dwork* Ravi Kumar† Moni Naor‡ D. Sivakumar§

ABSTRACT

We consider the problem of combining ranking results from various sources. In the context of the Web, the main applications include building meta-search engines, combining ranking functions, selecting documents based on multiple criteria, and improving search precision through word associations. We develop a set of techniques for the rank aggregation problem and compare their performance to that of well-known methods. A primary goal of our work is to design rank aggregation techniques that can effectively combat "spam," a serious problem in Web searches. Experiments show that our methods are simple, efficient, and effective.

Keywords: rank aggregation, ranking functions, meta-search, multi-word queries, spam

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

input

1	5	4	8	10	2	6	9	3	7

count inversions in left half A

1	5	4	8	10

5-4

count inversions in right half B

2	6	9	3	7

6-3 9-3 9-7

count inversions (a, b) with a ∈ A and b ∈ B

1	5	4	8	10	2	6	9	3	7

4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10-3 10-6 10-7 10-9

output 1 + 3 + 13 = 17

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element $b \in B$,
  - binary search in A to find how elements in A are greater than b.

list A

7	10	18	3	14

list B

17	23	2	11	16

sort A

3	7	10	14	18

sort B

2	11	16	17	23

binary search to count inversions (a, b) with a ∈ A and b ∈ B

3	7	10	14	18

2	11	16	17	23
5	2	1	1	0

Counting inversions: how to combine two subproblems?

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

count inversions (a, b) with a ∈ A and b ∈ B

3	7	10	a_i	18

2	11	b_j	17	23
5	2			

merge to form sorted list C

2	3	7	10	11					

Counting inversions: divide-and-conquer algorithm implementation

Input. List L.

Output. Number of inversions in L and sorted list of elements L'.

SORT-AND-COUNT (L)

IF list L has one element

  RETURN $(0, L)$.

DIVIDE the list into two halves A and B.

$(r_A, A) \leftarrow$ SORT-AND-COUNT(A).

$(r_B, B) \leftarrow$ SORT-AND-COUNT(B).

$(r_{AB}, L') \leftarrow$ MERGE-AND-COUNT(A, B).

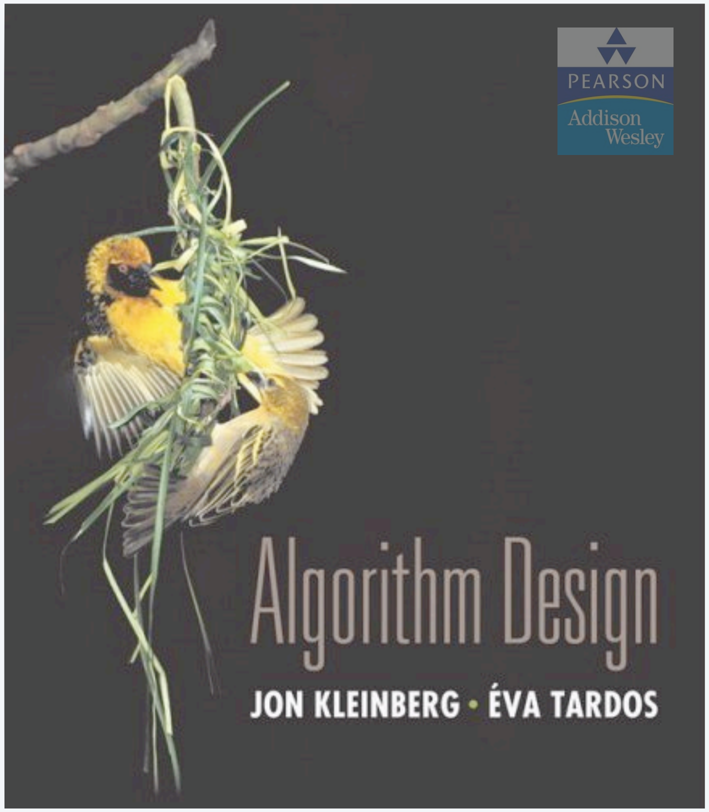
RETURN $(r_A + r_B + r_{AB}, L')$.

Counting inversions: divide-and-conquer algorithm analysis

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Pf. The worst-case running time $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$



SECTION 5.4

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*

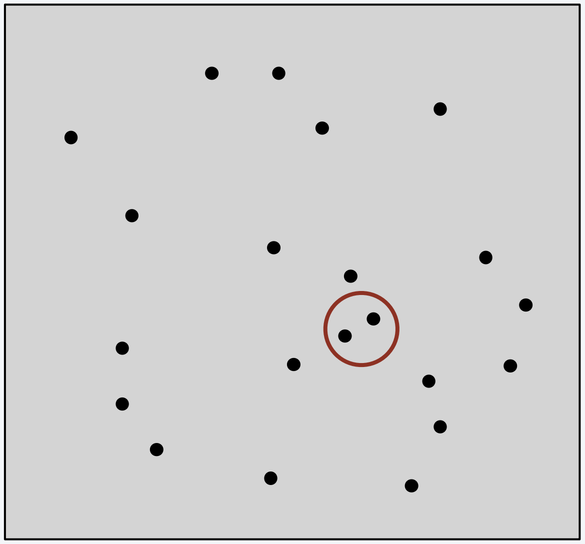
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems



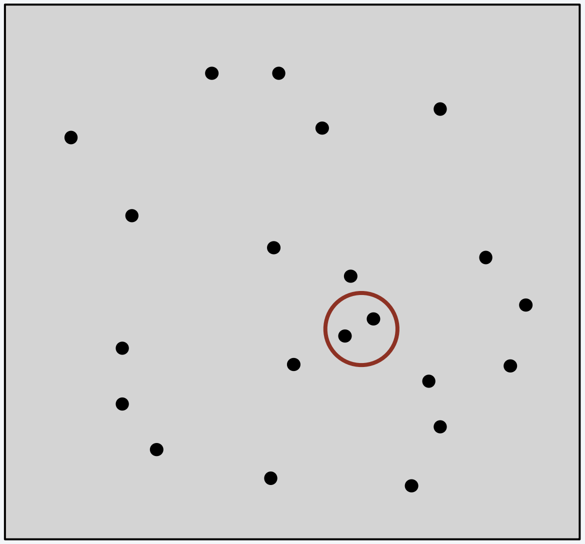
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.

1d version. Easy $O(n \log n)$ algorithm if points are on a line.

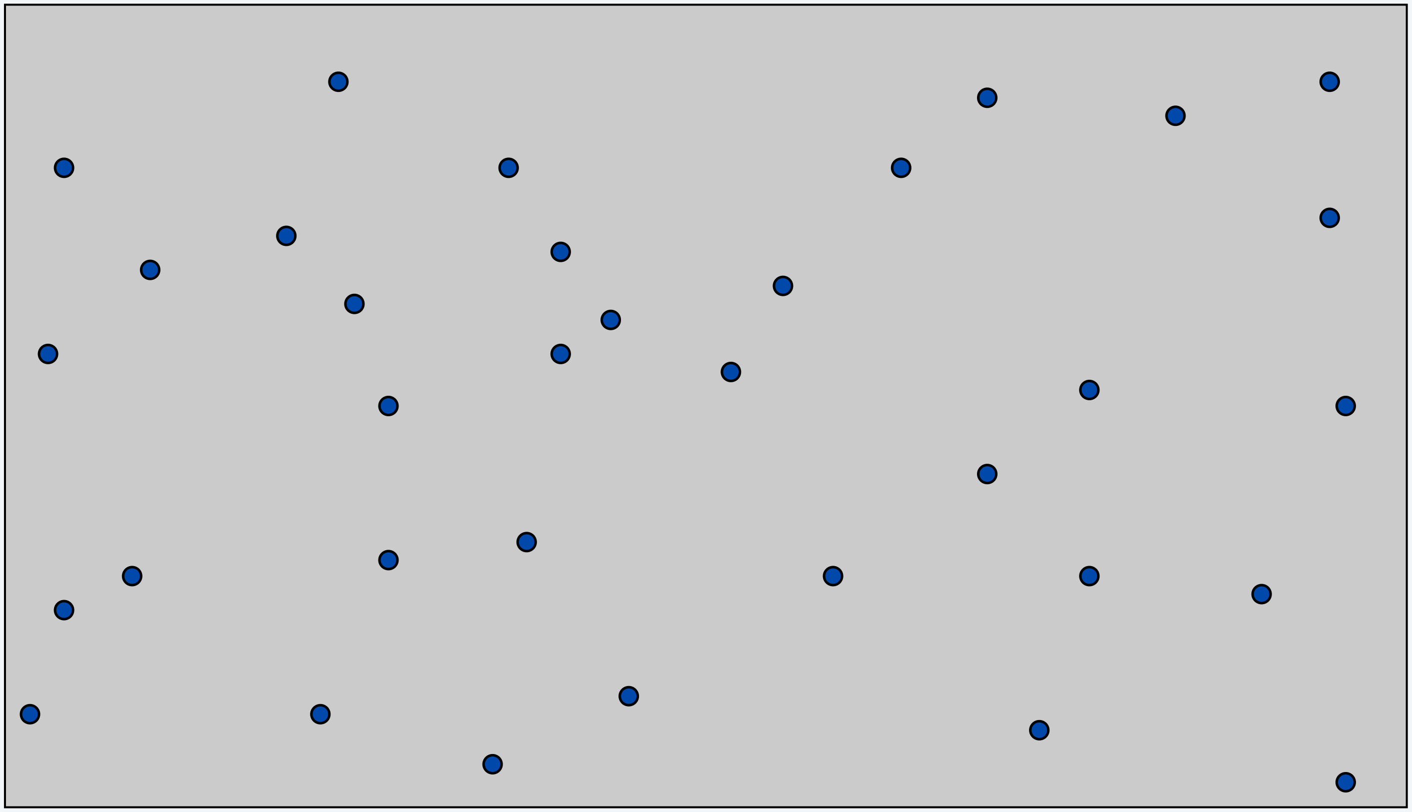
Nondegeneracy assumption. No two points have the same x-coordinate.



Closest pair of points: first attempt

Sorting solution.

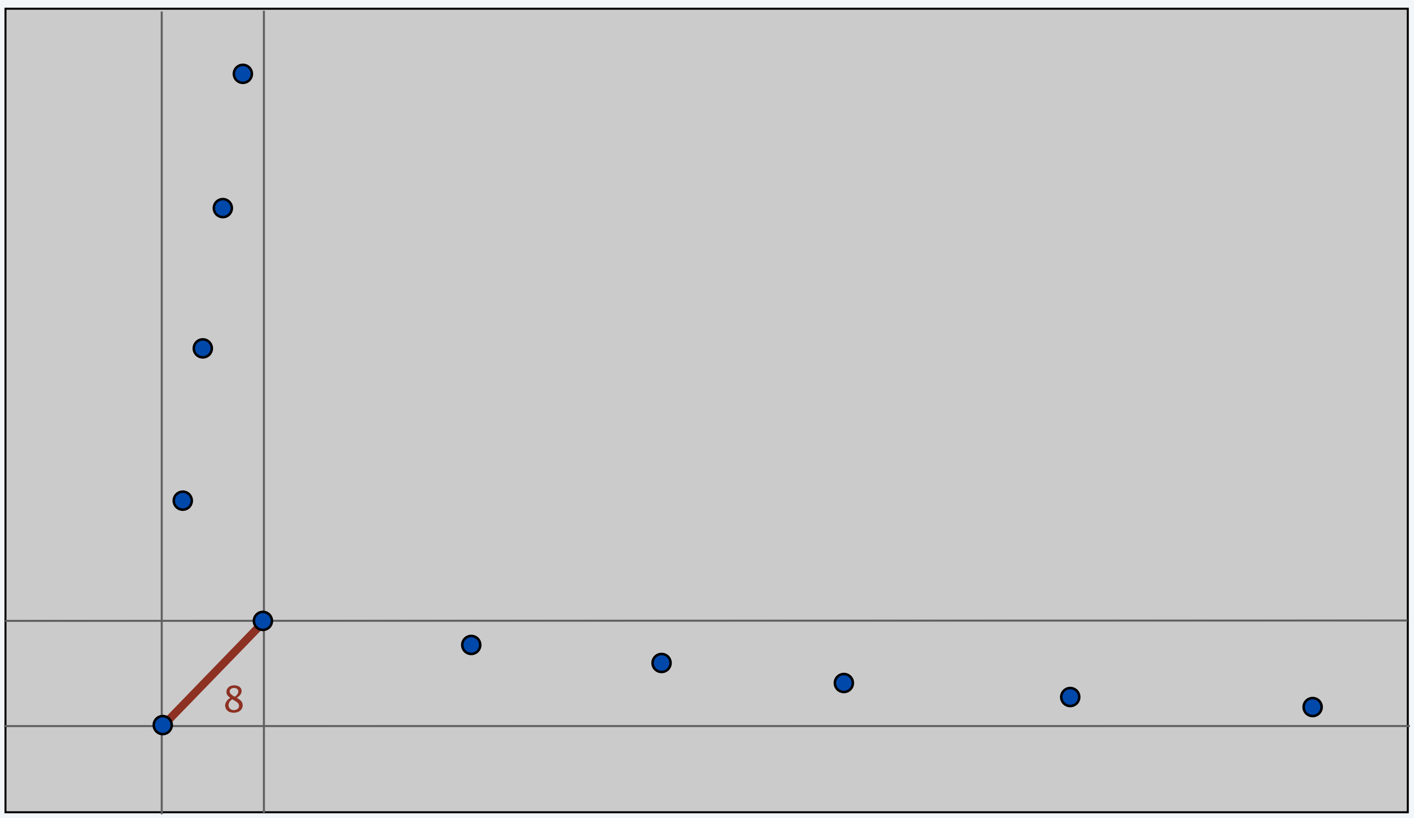
- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: first attempt

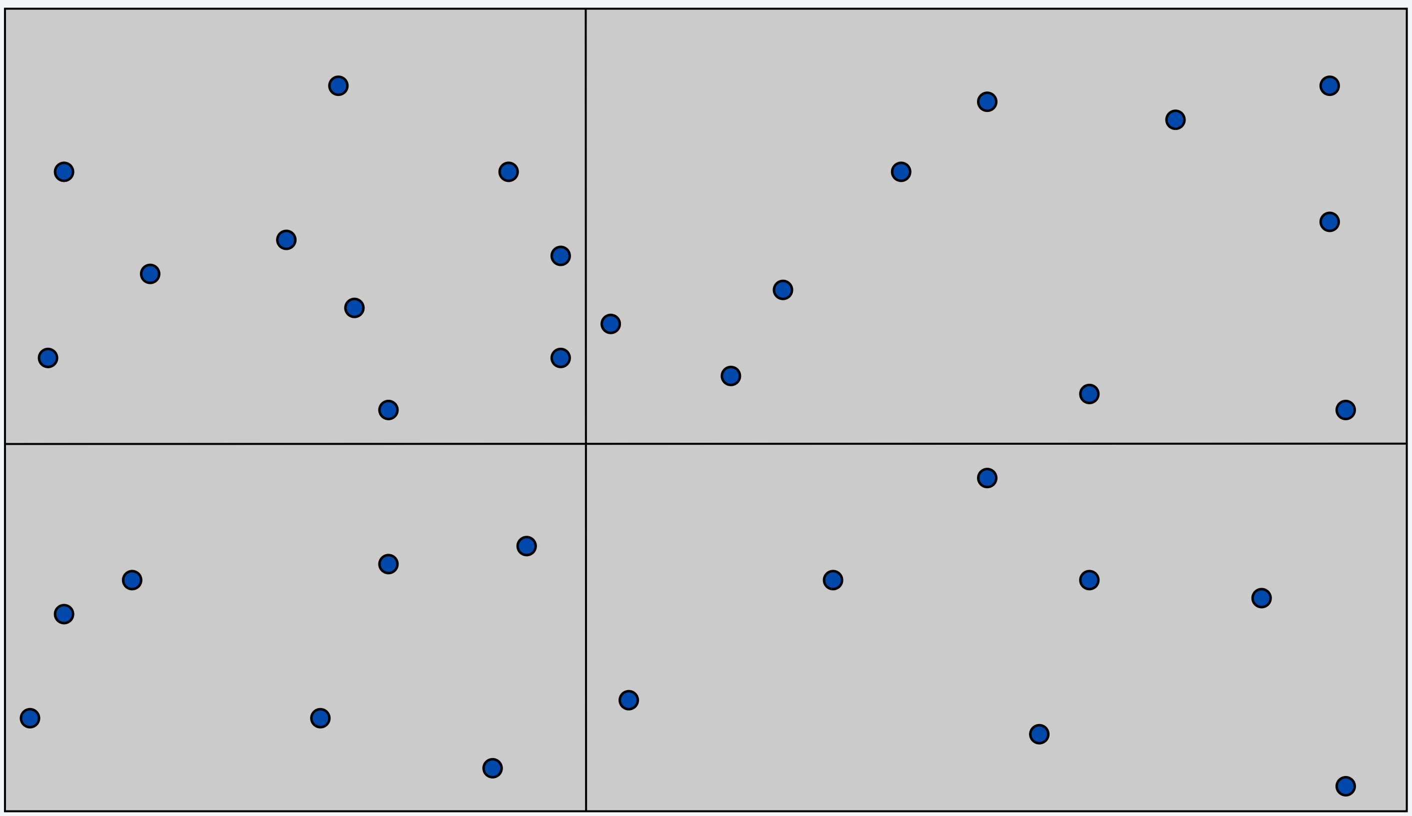
Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: second attempt

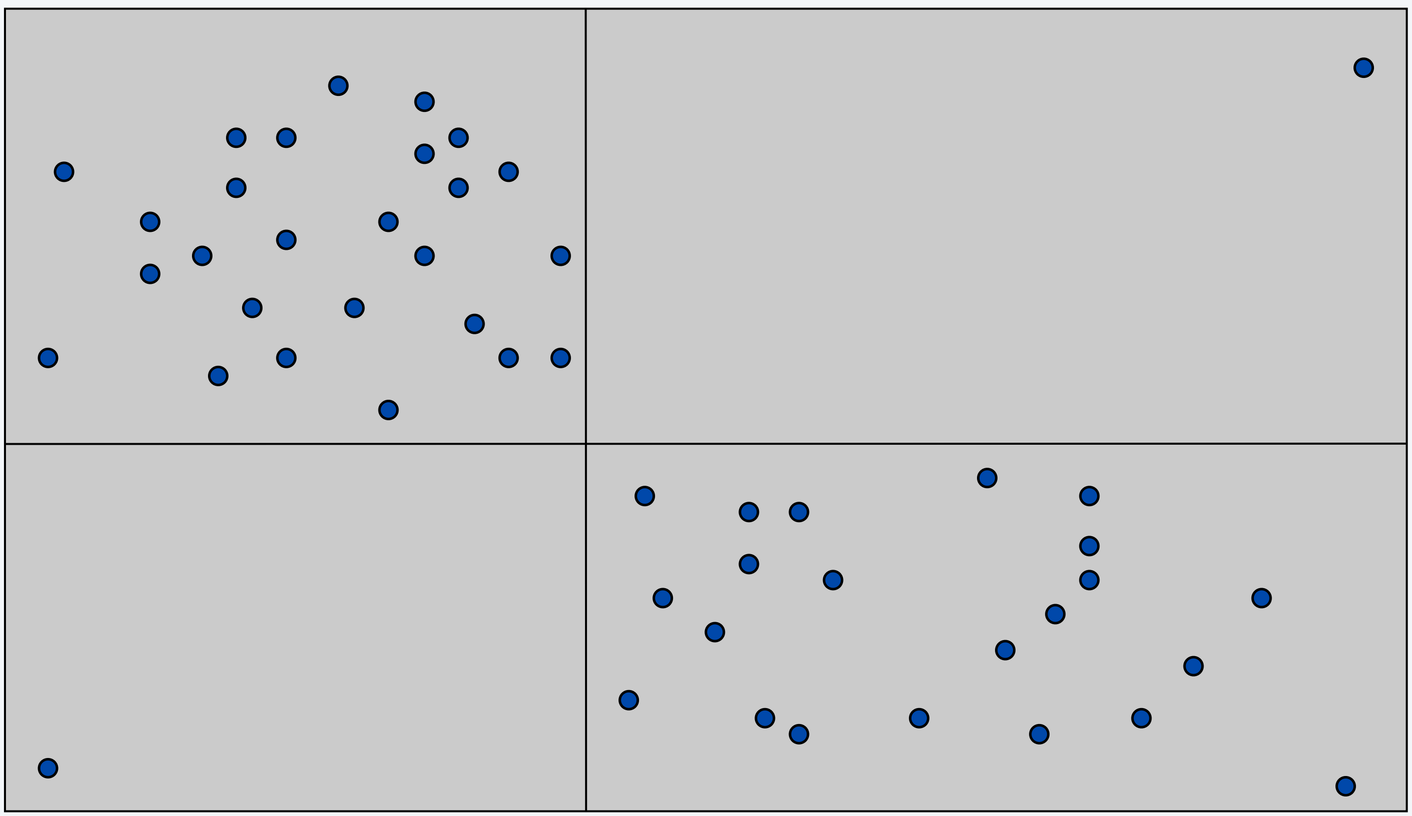
Divide. Subdivide region into 4 quadrants.



Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.

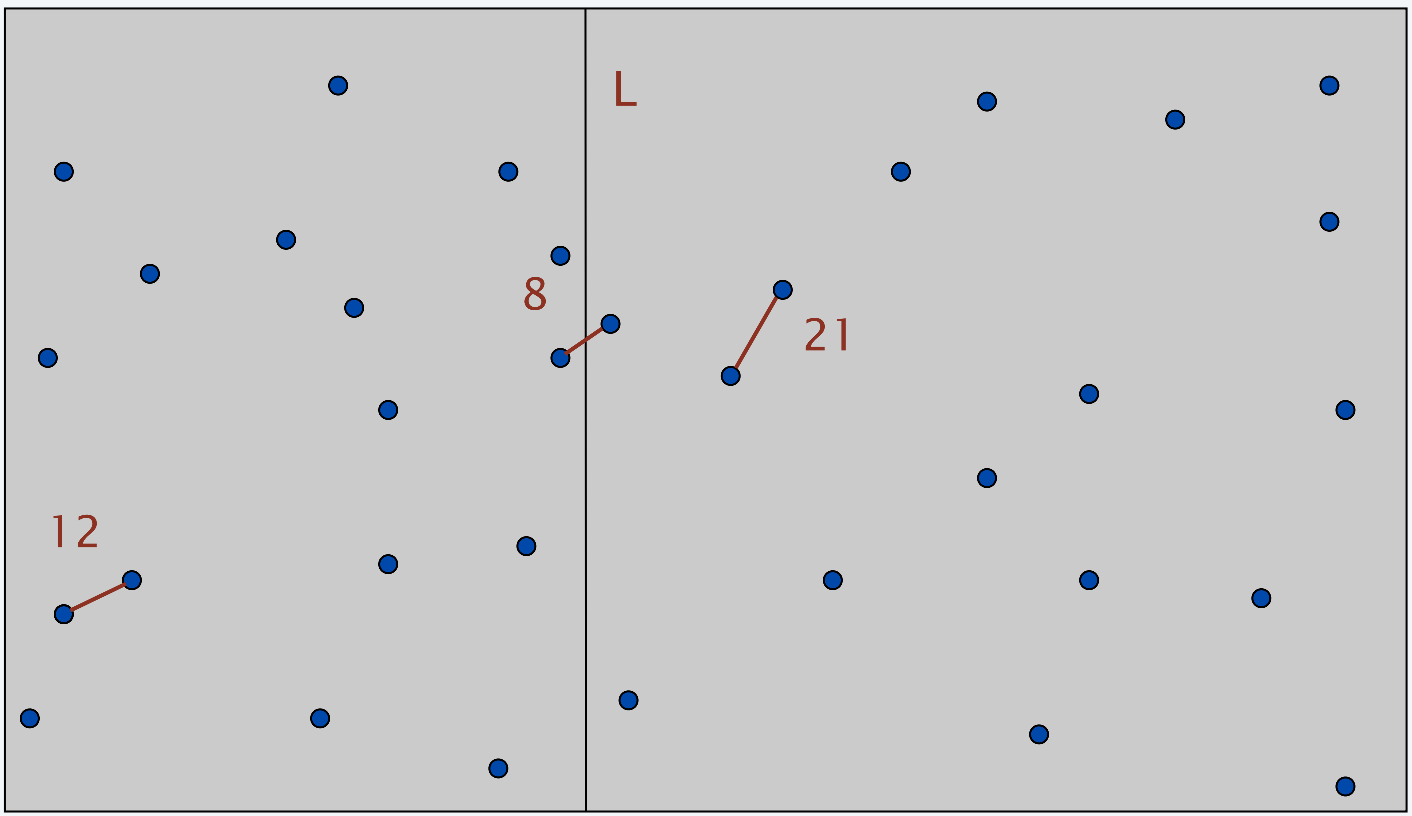
Obstacle. Impossible to ensure $n/4$ points in each piece.



Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

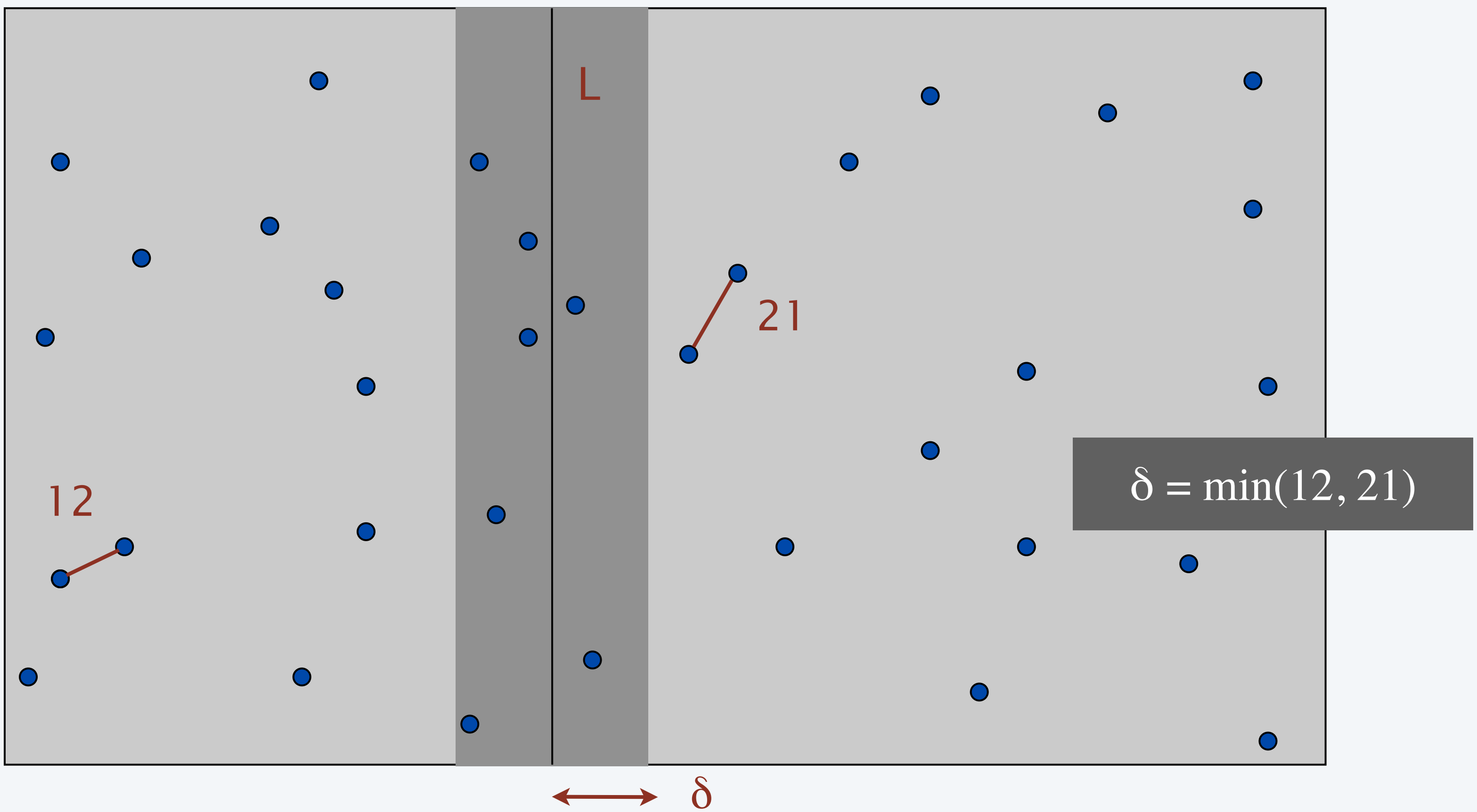
seems like $\Theta(N^2)$



How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.

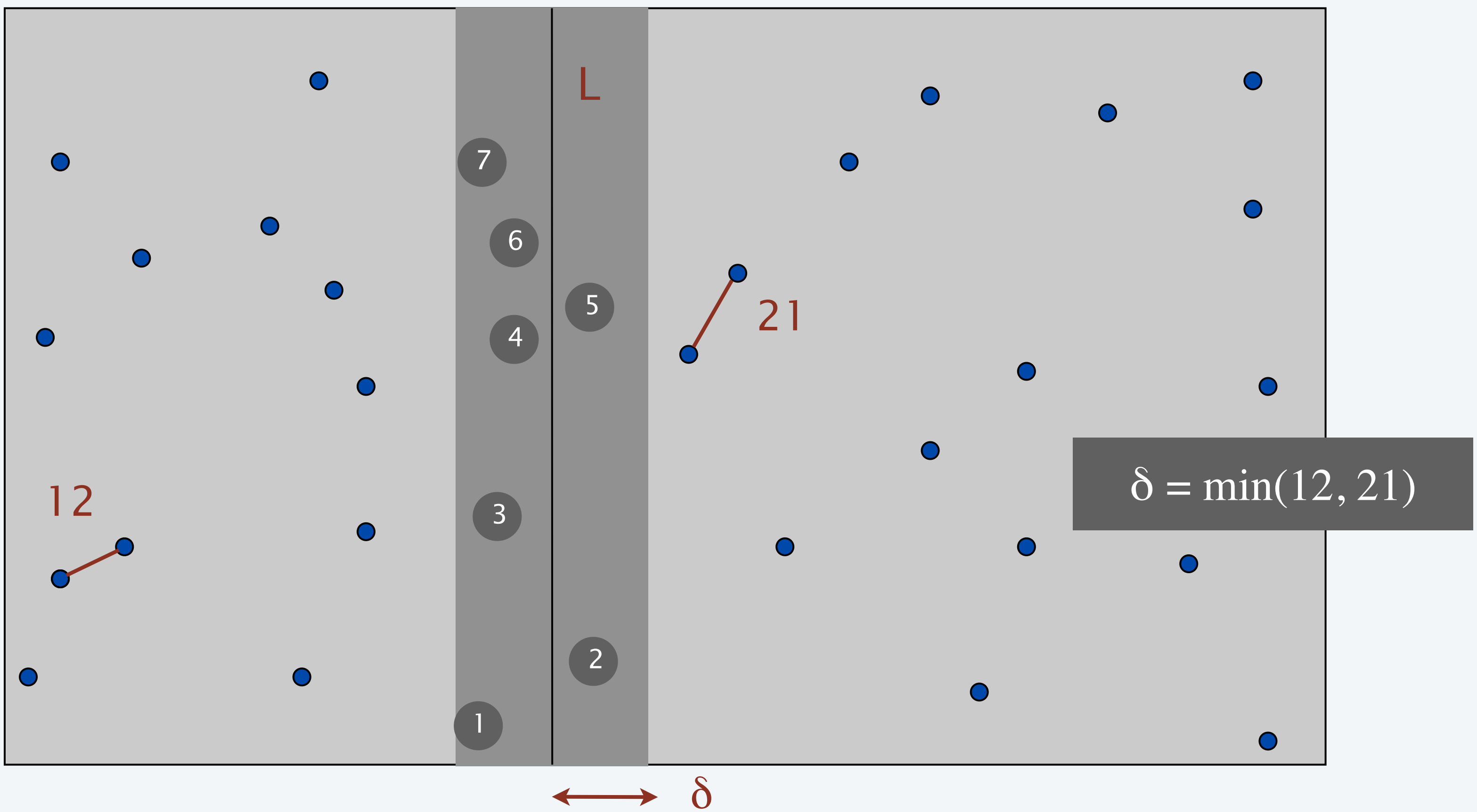


How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y-coordinate.
- Only check distances of those within 11 positions in sorted list!

why 11?



How to find closest pair with one point in each side?

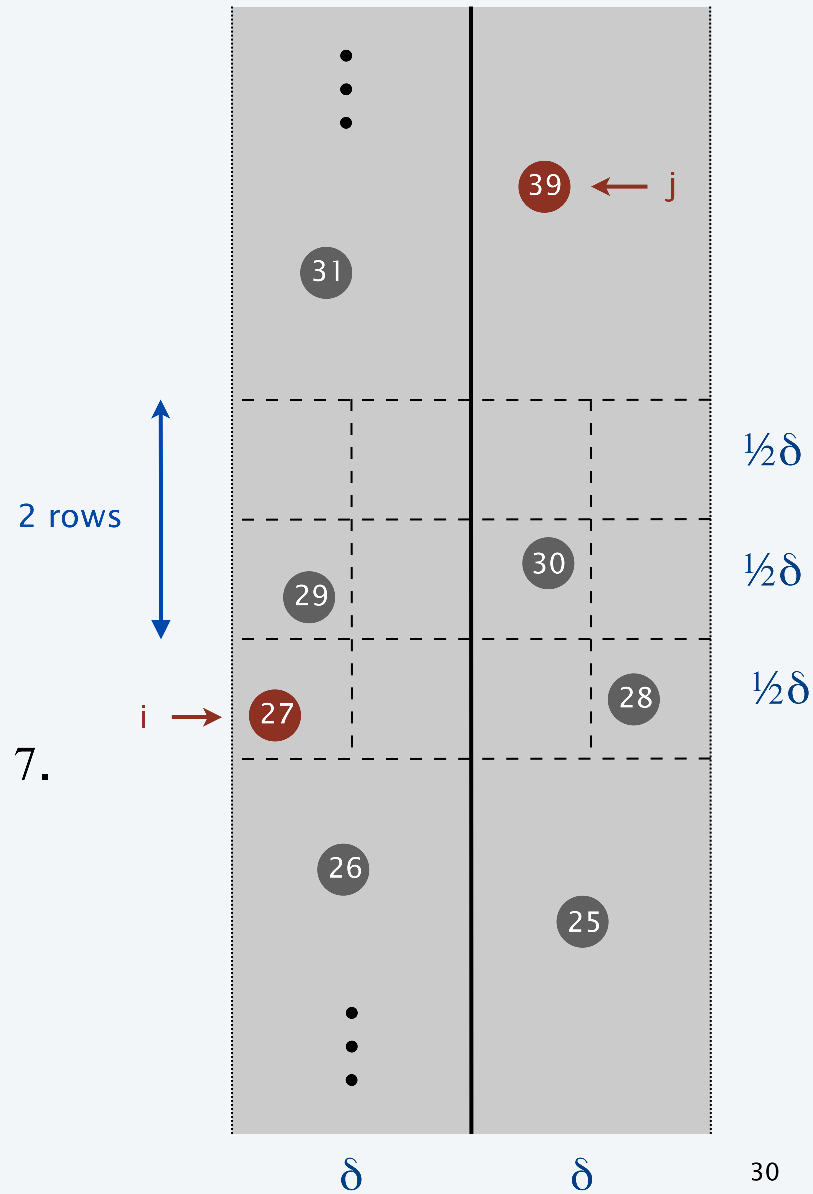
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Claim remains true if we replace 12 with 7.



Closest pair of points: divide-and-conquer algorithm

CLOSEST-PAIR $(p_1, p_2, \ldots, p_n)$

Compute separation line L such that half the points are on each side of the line. ← $O(n \log n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR (points in left half).

$\delta_2 \leftarrow$ CLOSEST-PAIR (points in right half). ← $2\,T(n / 2)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$.

Delete all points further than δ from line L. ← $O(n)$

Sort remaining points by y-coordinate. ← $O(n \log n)$

Scan points in y-order and compare distance between each point and next 11 neighbors. If any of these distances is less than δ, update δ. ← $O(n)$

RETURN δ.

Closest pair of points: analysis

Theorem. The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log^2 n)$ time.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n \log n) & \text{otherwise} \end{cases}$$

Improved closest pair algorithm

Q. How to improve to $O(n \log n)$?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by x-coordinate, and all points sorted by y-coordinate.
- Sort by merging two pre-sorted lists.

Theorem. [Shamos 1975] The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log n)$ time.

Pf.
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Note. See SECTION 13.7 for a randomized $O(n)$ time algorithm.

not subject to lower bound
since it uses the floor function