# Shortest paths: dynamic programming & Bellman Ford

Slides by Joseph Paul Cohen

Shortest paths: dynamic programming, O(mn) time and O(mn) space
Computing the shortest paths to a target t

```
For v ∈ V
  M[0, v] = ∞
M[0, t] = 0

For i = 1 : |V|-1
  For v ∈ V
    M[i, v] = M [i – 1, v]
    For (v, w) ∈ E
      M [i, v] = min{M [i, v]
                     M [i–1, w] + c_vw }
```

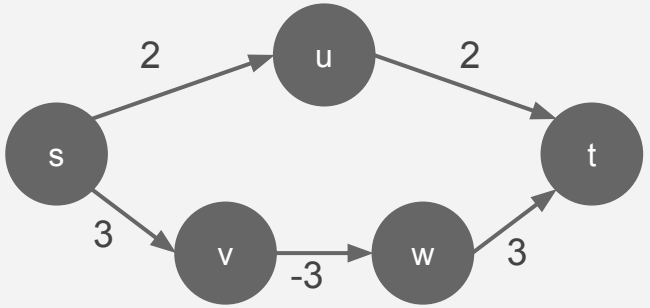
Initial distances from target

The longest simple path would include only |V|-1 edges

Value copied so M[i, v] will track the min for the iteration

After iterating through every edge our choice for M [i, v] is optimal

# Shortest paths: dynamic programming




```
min{M[2, w], M[1, t] + c_wt}
min{3, 3}
```

```
min{M[1, u], M[0, t] + c_ut}
min{∞, 0 + 2}
```

```
min{M[2, v], M[1, w] + c_vw}
min{∞, 3 + (-3)}
```

```
min{M[3, s], M[2, v] + c_sv}
min{4, 0 + 3}
```

| M | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **0** | ∞ | ∞ | ∞ | ∞ | 0 |
| **1** | ∞ | 2 | ∞ | 3 | 0 |
| **2** | 4 | 2 | 0 | 3 | 0 |
| **3** | 3 | 2 | 0 | 3 | 0 |
| **4** | 3 | 2 | 0 | 3 | 0 |

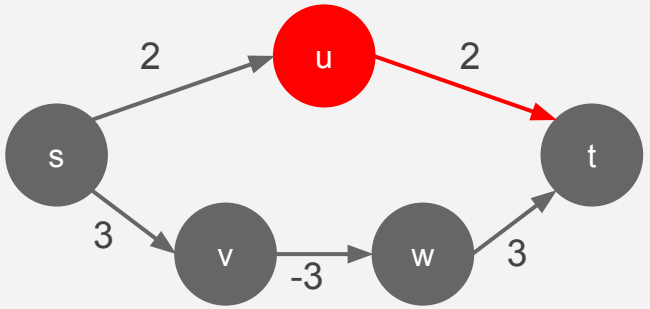
# Shortest paths: dynamic programming

$\min\{M[2, w], M[1, t] + c_{wt}\}$
$\min\{3, 3\}$

$\min\{M[1, u], M[0, t] + c_{ut}\}$
$\min\{\infty, 0 + 2\}$

$\min\{M[2, v], M[1, w] + c_{vw}\}$
$\min\{\infty, 3 + (-3)\}$

$\min\{M[3, s], M[2, v] + c_{sv}\}$
$\min\{4, 0 + 3\}$




| M | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **0** | ∞ | ∞ | ∞ | ∞ | 0 |
| **1** | ∞ | 2 | ∞ | 3 | 0 |
| **2** | 4 | 2 | 0 | 3 | 0 |
| **3** | 3 | 2 | 0 | 3 | 0 |
| **4** | 3 | 2 | 0 | 3 | 0 |

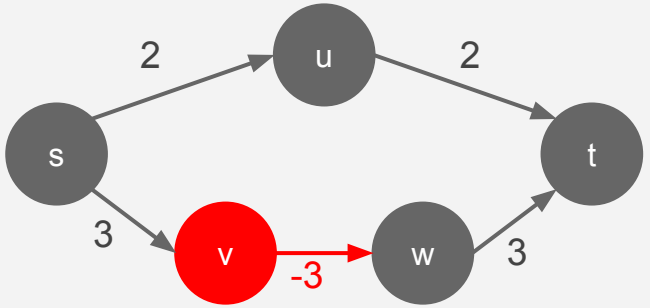
# Shortest paths: dynamic programming

$\min\{M[2, w], M[1, t] + c_{wt}\}$
$\min\{3, 3\}$

$\min\{M[1, u], M[0, t] + c_{ut}\}$
$\min\{\infty, 0 + 2\}$

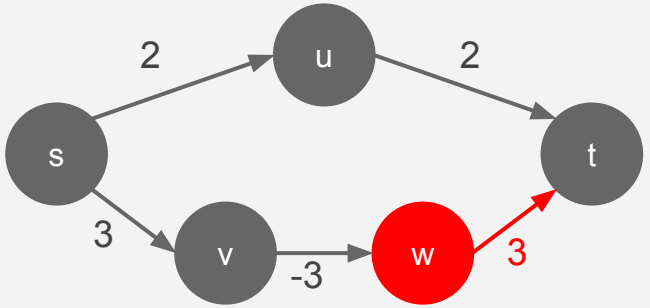
$\min\{M[2, v], M[1, w] + c_{vw}\}$
$\min\{\infty, 3 + (-3)\}$

$\min\{M[3, s], M[2, v] + c_{sv}\}$
$\min\{4, 0 + 3\}$




| M | s | u | v | w | t |
|---|---|---|---|---|---|
| **0** | ∞ | ∞ | ∞ | ∞ | 0 |
| **1** | ∞ | 2 | ∞ | 3 | 0 |
| **2** | 4 | 2 | 0 | 3 | 0 |
| **3** | 3 | 2 | 0 | 3 | 0 |
| **4** | 3 | 2 | 0 | 3 | 0 |

# Shortest paths: dynamic programming




```
min{M[2, w], M[1, t] + c_wt}
min{3, 3}
```

```
min{M[1, u], M[0, t] + c_ut}
min{∞, 0 + 2}
```

```
min{M[2, v], M[1, w] + c_vw}
min{∞, 3 + (-3)}
```

```
min{M[3, s], M[2, v] + c_sv}
min{4, 0 + 3}
```

| M | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **0** | ∞ | ∞ | ∞ | ∞ | 0 |
| **1** | ∞ | 2 | ∞ | 3 | 0 |
| **2** | 4 | 2 | 0 | 3 | 0 |
| **3** | 3 | 2 | 0 | 3 | 0 |
| **4** | 3 | 2 | 0 | 3 | 0 |

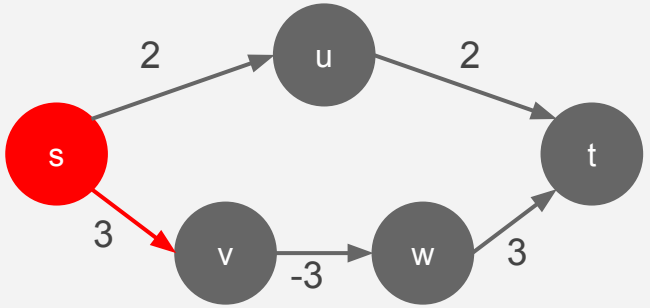
# Shortest paths: dynamic programming

$\min\{M[2, w], M[1, t] + c_{wt}\}$
$\min\{3, 3\}$

$\min\{M[1, u], M[0, t] + c_{ut}\}$
$\min\{\infty, 0 + 2\}$

$\min\{M[2, v], M[1, w] + c_{vw}\}$
$\min\{\infty, 3 + (-3)\}$

$\min\{M[3, s], M[2, v] + c_{sv}\}$
$\min\{4, 0 + 3\}$




| M | s | u | v | w | t |
|---|---|---|---|---|---|
| **0** | ∞ | ∞ | ∞ | ∞ | 0 |
| **1** | ∞ | 2 | ∞ | 3 | 0 |
| **2** | 4 | 2 | 0 | 3 | 0 |
| **3** | 3 | 2 | 0 | 3 | 0 |
| **4** | 3 | 2 | 0 | 3 | 0 |

# Shortest paths:
# **Bellman Ford**

Slides by Joseph Paul Cohen

# Shortest paths: Bellman Ford, O(mn) time and O(n) space

Direction: shortest paths to target `t`

```
For v ∈ V
  dist[v] = ∞
  edge[v] = null
dist[t] = 0

For i = 1 : |V|-1
  For (v, w) ∈ E
    If dist[v] > dist[w] + c_vw
      dist[v] = dist[w] + c_vw
      edge[v] = w
```

Initial distances

dist[t] = 0: SP's to t

The longest simple path would include only |V|-1 edges

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

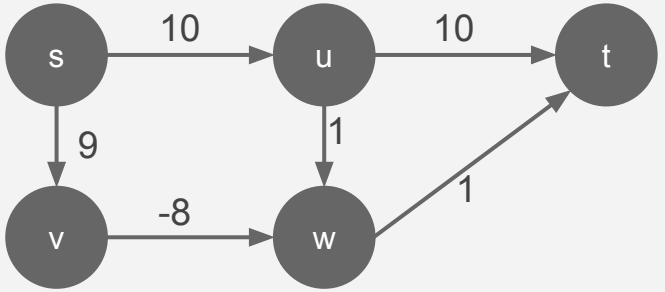
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=1** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

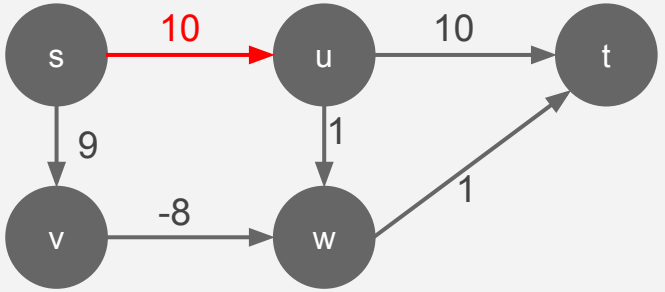
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=1** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

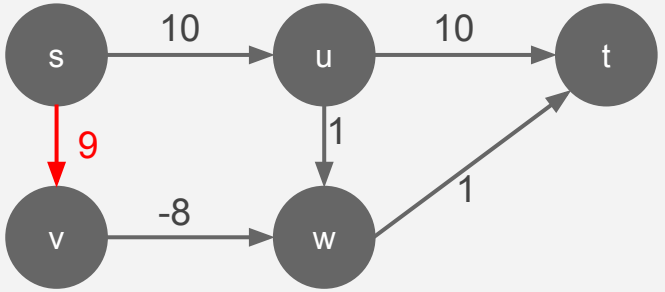
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=1** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

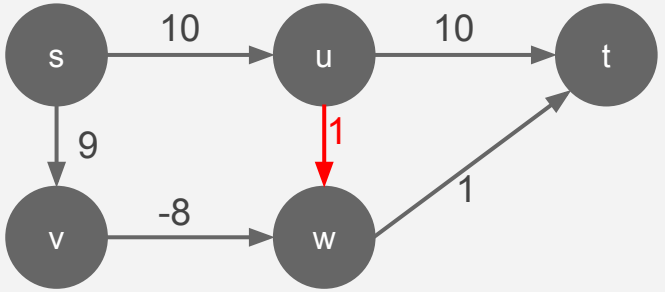
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=1** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

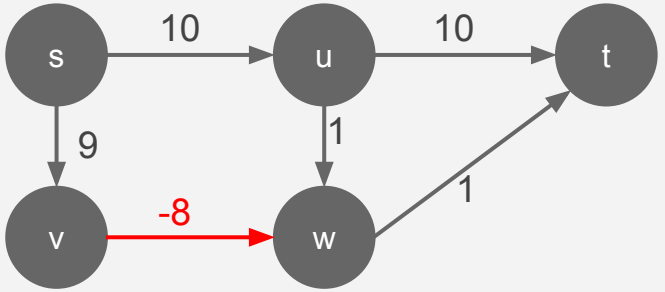
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | s | u | v | w | t |
|---|---|---|---|---|---|
| **i=0** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=1** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

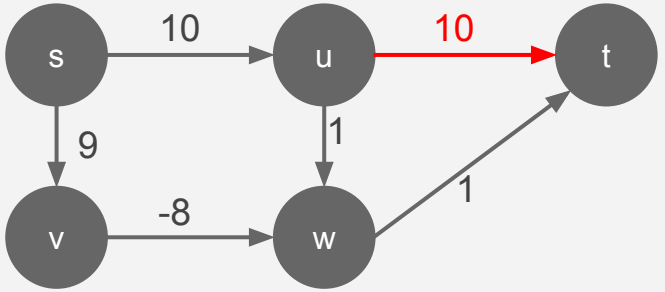
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,– | ∞,– | ∞,– | ∞,– | 0,t |
| **i=1** | ∞,– | 10,t | ∞,– | ∞,– | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

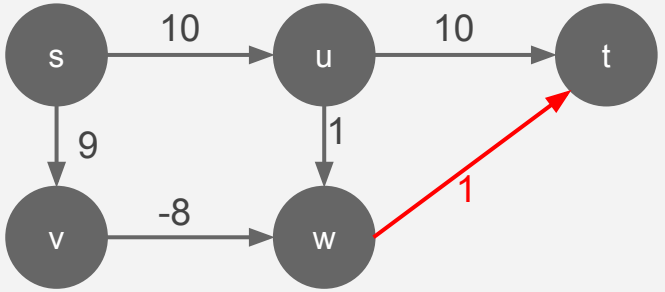
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,- | ∞,- | ∞,- | ∞,- | 0,t |
| **i=1** | ∞,- | 10,t | ∞,- | 1,t | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    ∞    + 10    = FALSE
```

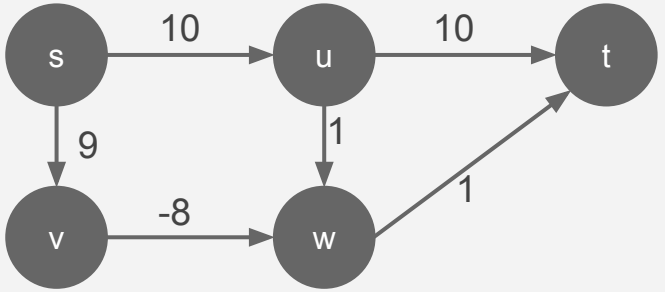
```
dist[s] > dist[v] + c_sv
   ∞    >    ∞    + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   ∞    >    ∞    + 1     = FALSE
```

```
dist[v] > dist[w] + c_su
   ∞    >    ∞    + -8    = FALSE
```

```
dist[u] > dist[t] + c_ut
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = t
```

```
dist[w] > dist[t] + c_wt
   10   >    0    + 1     = TRUE
dist[w] = 1, edge[w] = t
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,- | ∞,- | ∞,- | ∞,- | 0,t |
| **i=1** | ∞,- | 10,t | ∞,- | 1,t | 0,t |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   ∞    >    10    + 10    = TRUE
dist[s] = 20, edge[s] = u
```

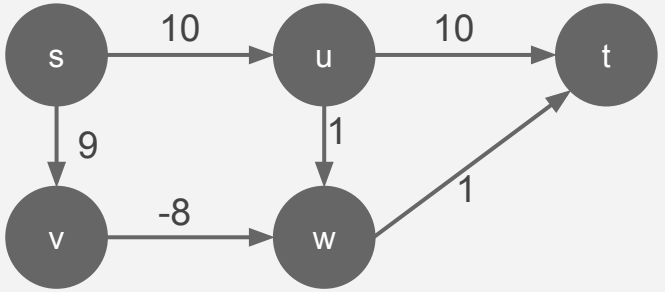
```
dist[s] > dist[v] + c_sv
   20   >    ∞     + 9     = FALSE
```

```
dist[u] > dist[w] + c_uw
   10   >    1     + 1     = TRUE
dist[u] = 2, edge[u] = w
```

```
dist[v] > dist[w] + c_su
   ∞    >    1     + -8    = TRUE
dist[v] = -7, edge[v] = w
```

```
dist[u] > dist[t] + c_ut
   2    >    0     + 10    = FALSE
```

```
dist[w] > dist[t] + c_wt
   1    >    0     + 1     = FALSE
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,- | ∞,- | ∞,- | ∞,- | 0,t |
| **i=1** | ∞,- | 10,t | ∞,- | 1,t | 0,t |
| **i=2** | 20,u | 2,w | -7,w | 1,t | 0,t |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   20   >   10    + 10     = FALSE
```

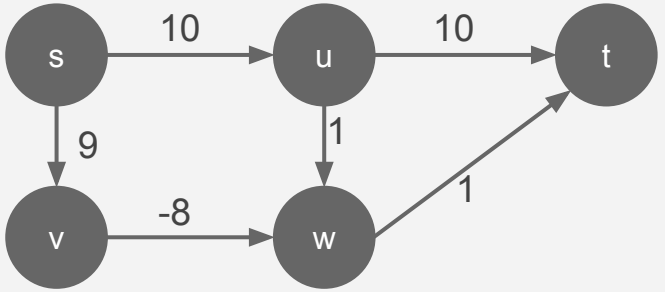
```
dist[s] > dist[v] + c_sv
   20   >   -7    + 9      = TRUE
dist[s] = 2, edge[s] = v
```

```
dist[u] > dist[w] + c_uw
   2    >    1    + 1      = FALSE
```

```
dist[v] > dist[w] + c_su
   -7   >    1    + -8     = FALSE
```

```
dist[u] > dist[t] + c_ut
   2    >    0    + 10     = FALSE
```

```
dist[w] > dist[t] + c_wt
   1    >    0    + 1      = FALSE
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,- | ∞,- | ∞,- | ∞,- | 0,t |
| **i=1** | ∞,- | 10,t | ∞,- | 1,t | 0,t |
| **i=2** | 20,u | 2,w | -7,w | 1,t | 0,t |
| **i=3** | 2,v | 2,w | -7,w | 1,t | 0,t |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[s] > dist[u] + c_su
   2    >   10    + 10     = FALSE
```

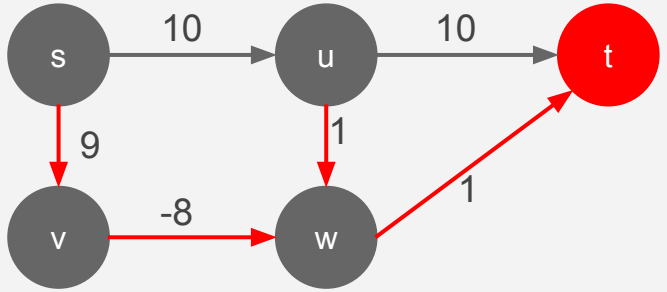
```
dist[s] > dist[v] + c_sv
   2    >   -7    + 9      = FALSE
```

```
dist[u] > dist[w] + c_uw
   2    >    1    + 1      = FALSE
```

```
dist[v] > dist[w] + c_su
   -7   >    1    + -8     = FALSE
```

```
dist[u] > dist[t] + c_ut
   2    >    0    + 10     = FALSE
```

```
dist[w] > dist[t] + c_wt
   1    >    0    + 1      = FALSE
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | ∞,- | ∞,- | ∞,- | ∞,- | 0,t |
| **i=1** | ∞,- | 10,t | ∞,- | 1,t | 0,t |
| **i=2** | 20,u | 2,w | -7,w | 1,t | 0,t |
| **i=3** | 2,v | 2,w | -7,w | 1,t | 0,t |
| **i=4** | 2,v | 2,w | -7,w | 1,t | 0,t |

# Shortest paths: Bellman Ford, O(mn) time and O(n) space

Direction: shortest paths from source s

```
For v ∈ V
  dist[v] = ∞
  edge[v] = null
dist[s] = 0

For i = 1 : |V|-1
  For (v, w) ∈ E
    If dist[w] > dist[v] + c_vw
      dist[w] = dist[v] + c_vw
      edge[w] = v
```

Initial distances

dist[s] = 0: SP's from s

The longest simple path would include only |V|-1 edges

# Shortest paths: dynamic programming

```
dist[u] > dist[s] + c_su
   ∞    >    0    + 10    = TRUE
dist[u] = 10, edge[u] = s
```

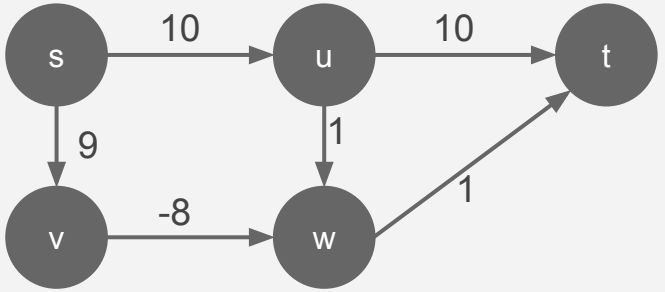
```
dist[v] > dist[s] + c_sv
   ∞    >    0    + 9     = TRUE
dist[v] = 9, edge[v] = s
```

```
dist[w] > dist[u] + c_uw
   ∞    >    10   + 1     = TRUE
dist[w] = 11, edge[w] = u
```

```
dist[w] > dist[v] + c_vw
   ∞    >    9    + -8    = TRUE
dist[w] = 1, edge[w] = v
```

```
dist[t] > dist[u] + c_ut
   ∞    >    10   + 10    = TRUE
dist[t] = 20, edge[t] = u
```

```
dist[t] > dist[w] + c_wt
   ∞    >    1    + 1     = TRUE
dist[t] = 2, edge[t] = w
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | 0,s | ∞,- | ∞,- | ∞,- | ∞,- |
| **i=1** | 0,s | 10,s | 9,s | 1,v | 2,w |
| **i=2** | | | | | |
| **i=3** | | | | | |
| **i=4** | | | | | |

# Shortest paths: dynamic programming

```
dist[u] > dist[s] + c_su
   10   >    0    + 10    = FALSE
```

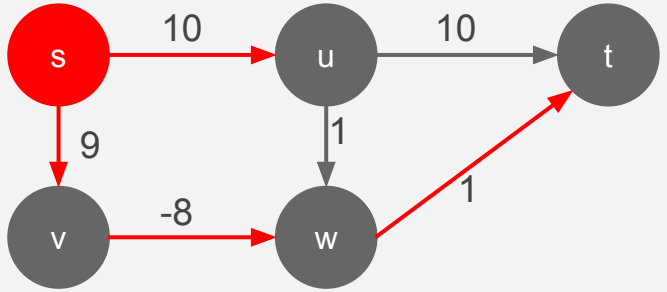
```
dist[v] > dist[s] + c_sv
   9    >    0    + 9     = FALSE
```

```
dist[w] > dist[u] + c_uw
   1    >    10   + 1     = FALSE
```

```
dist[w] > dist[v] + c_vw
   1    >    9    + -8    = FALSE
```

```
dist[t] > dist[u] + c_ut
   2    >    10   + 10    = FALSE
```

```
dist[t] > dist[w] + c_wt
   2    >    1    + 1     = FALSE
```




edges:
s->u=10
s->v=9
u->w=1
v->w=-8
u->t=10
w->t=1

| dist, edge | **s** | **u** | **v** | **w** | **t** |
|---|---|---|---|---|---|
| **i=0** | 0,s | ∞,- | ∞,- | ∞,- | ∞,- |
| **i=1** | 0,s | 10,s | 9,s | 1,v | 2,w |
| **i=2** | 0,s | 10,s | 9,s | 1,v | 2,w |
| **i=3** | | | | | |
| **i=4** | | | | | |